A NOTE ON THE MUMFORD-TATE CONJECTURE FOR CM ABELIAN VARIETIES

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Abstract. The Mumford-Tate conjecture is first proved for CM abelian varieties by H. Pohlmann [Ann. Math., 1968]. In this note we give another proof of this result and extend it to CM motives.

1. Introduction

Let A be an abelian variety over a field k, where k is finitely generated over $\mathbb{Q}$ and is contained in the field $\mathbb{C}$ of complex numbers. The Mumford-Tate group $MT(A)$ of $A_{\mathbb{C}} := A \otimes \mathbb{C}$ is defined to be the smallest algebraic $\mathbb{Q}$-subgroup G of $\mathrm{GL}(V)$ such that $G(\mathbb{C})$ contains the image of the Hodge cocharacter $\mu : \mathbb{C}^\times \to \mathrm{GL}(V_{\mathbb{C}})$, where $V := H_1(A(\mathbb{C}), \mathbb{Q})$ is the first rational homology group of A and $V_{\mathbb{C}} = V \otimes \mathbb{C}$. For any rational prime ℓ, let $G_{A,\ell}$ denote the algebraic envelope of the associated ℓ-adic Galois representation

$$\rho_\ell : \mathrm{Gal}(\bar{k}/k) \to \mathrm{GL}(V_\ell(A)),$$

where $V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, $T_\ell(A)$ is the ℓ-adic Tate module of A and $\bar{k}$ is the algebraic closure of k in $\mathbb{C}$. That is, $G_{A,\ell}$ is the Zariski closure of the image $\rho_\ell(\mathrm{Gal}(\bar{k}/k))$ in the algebraic $\mathbb{Q}_\ell$-group $\mathrm{GL}(V_\ell(A))$. Let $G^0_{A,\ell}$ denote the neutral component of the algebraic group $G_{A,\ell}$ over $\mathbb{Q}_\ell$.

Under the comparison isomorphism $V_\ell := V \otimes \mathbb{Q}_\ell \simeq V_\ell(A)$ and hence $\mathrm{GL}(V) \otimes \mathbb{Q}_\ell$ being identified with the $\mathbb{Q}_\ell$-group $\mathrm{GL}(V_\ell(A))$, the Mumford-Tate conjecture (MTC) asserts the equality of these two algebraic subgroups

$$MT(A) \otimes \mathbb{Q}_\ell = G^0_{A,\ell} \tag{1.1}$$

of $\mathrm{GL}(V_\ell)$ for all primes ℓ.

Observe that the group $G^0_{A,\ell}$ is unchanged if one replaces the field k by a finitely generated field extension k_1 of k in $\mathbb{C}$. Indeed, as one has the natural identification $T_\ell(A \otimes \bar{k}) = T_\ell(A \otimes \bar{k}_1)$, the action of the Galois group $\mathrm{Gal}(\bar{k}_1/k_1)$ on the Tate module factors through its quotient $\mathrm{Gal}(\bar{k}/\bar{k} \cap k_1)$. Therefore, the group $G^0_{A,\ell}$ is unchanged if one replaces A by $A \otimes_k k_1$ since it remains the same after a finite field extension base change. In particular, the MTC does not depend on the choice of the finitely generated field k.

The meaning of the MTC asserts that the algebra of all ℓ-adic Tate cohomology classes (with respect to all finite algebraic extensions of k) on every self-product A^m of A coincides with the $\mathbb{Q}_\ell$-algebra generated by Hodge classes of $A_{\mathbb{C}}$. For

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CM abelian varieties, this statement is proved by H. Pohlmann [9, Theorem 5] and hence the MTC follows.

In this expository article we give another proof of the MTC for abelian varieties of CM type. Observe that both groups concerned are subtori of the algebraic torus associated to the CM field in question (also see Section 2.2). We prove the equality by checking their co-character groups. The proof is based on the Taniyama-Shimura theory of complex multiplication of abelian varieties ([18] also see [16] and [17]), which is the same as used in Pohlmann's proof. Therefore, we do not claim any novelty in this expository article. Note that although [9] is the main reference where the first case of MTC is proved, the word "Mumford-Tate" does not appear in the paper. We also found the review article by Ribet [13, p. 216] where he pointed out that this is a corollary of the main theorem of complex multiplication due to Shimura-Taniyama [18] (but without referring to Pohlmann's work). This is indeed the case, and we are simply adding more details to this "corollary" here for the reader's convenience.

The Mumford-Tate conjecture has been verified for a large class of important cases. It is proved by Deligne [4, I. Proposition 6.2] that the inclusion

$$G^0_{A,\ell} \subset MT(A) \otimes \mathbb{Q}_\ell$$

holds for all primes ℓ. Serre has established several fundamental tools and methods for analyzing ℓ-adic Galois representations (see [15] and references given there). Among other results he [14] proved the MTC for abelian varieties A with $\mathrm{End}_{\bar{k}}(A) = \mathbb{Z}$ and $\dim A = 2, 6$ or odd. Ribet has developed several methods and among other results he proved the MTC for abelian varieties with real multiplication (the generic case); see [10, 11, 12]. The centralizers of $G_{A,\ell}$ and of $MT(A) \otimes \mathbb{Q}_\ell$ coincide in $\mathrm{GL}(V_\ell)$; this follows from Faltings' theorem [5] for Tate's conjecture on homomorphisms of abelian varieties. The rank of $G_{A,\ell}$ (the dimension of a maximal subtorus) is independent of primes ℓ (see Serre [14, 2.2.4, p. 31]). If one has the equality $G^0_{A,\ell} = MT(A) \otimes \mathbb{Q}_\ell$ for one prime ℓ, then one has the quality $G^0_{A,\ell} = MT(A) \otimes \mathbb{Q}_\ell$ for all primes ℓ (due to Tankeev [19, 20], Larsen and Pink [8, 6]).

We remark that the Mumford-Tate group $MT(A)$ is the fundamental group of the Tannakian category generated by the Hodge structure $V = H_1(A_\mathbb{C}, \mathbb{Q})$ and the Tate twists $\mathbb{Q}(1)$ (see [4]). Similarly the group $G^0_{A,\ell}$ is the fundamental group of the Tannakian category generated by the $\rho_\ell(\mathrm{Gal}(\bar{k}/k_1))$-representation $V_\ell(A)$ and the Tate twist $\mathbb{Q}_\ell(1)$, where k_1 is any finite extension of k so that $\rho_\ell(\mathrm{Gal}(\bar{k}/k_1))$ is contained in $G^0_{A,\ell}(\mathbb{Q}_\ell)$. Thus, the equality (1.1) asserts the equivalence of these two Tannakian categories. Clearly these groups can be defined also for more general projective smooth varieties through the singular cohomology and etale cohomology, and hence the MTC can be formulated for more general varieties even for motives. In the last section, we show how the MTC for CM motives follows from that for CM abelian varieties.

2. Algebraic Tori

2.1. **Basic properties of k-tori.** Let k be any field of characteristic zero. Let $\Gamma_k := \mathrm{Gal}(\bar{k}/k)$ denote the absolute Galois group of k, where $\bar{k}$ is an algebraic closure of k. Let (Diag. groups/k) denote the category of diagonalizable groups over k (see [2]); this is an abelian category. Let (Tori/k) denote the category of algebraic tori over k; this is a full subcategory of (Diag. groups/k) but not an abelian category. Let ($\mathbb{Z}[\Gamma_k]$-mod) denote the abelian category of finitely generated $\mathbb{Z}$-modules X together with a continuous action of Γ_k (with the discrete topology on X and the Krull topology on Γ_k). Finally let (free $\mathbb{Z}[\Gamma_k]$-mod) denote the full subcategory consisting of X which are free as $\mathbb{Z}$-modules. To each diagonalizable group D over k, we associate the character group $X^*(D)$ and the cocharacter group $X_*(D)$ as follows:

$$X^*(D) := \mathrm{Hom}_{\bar{k}}(D, \mathbb{G}_{\mathrm{m}}), \quad X_*(D) := \mathrm{Hom}_{\bar{k}}(\mathbb{G}_{\mathrm{m}}, D).$$

These are finitely generated $\mathbb{Z}$-modules equipped with a continuous Γ_k-action.

We have the following basic results (see [2]).

Theorem 2.1.

(1) *The functor X^* gives rise to an anti-equivalence of abelian categories between* (Diag. groups/k) *and* ($\mathbb{Z}[\Gamma_k]$-mod) *which preserves the short exact sequences. Moreover, it induces an anti-equivalence between the categories* (Tori/k) *and* (free $\mathbb{Z}[\Gamma_k]$-mod).

(2) *The functor X_* gives rise to an equivalence of categories between* (Tori/k) *and* (free $\mathbb{Z}[\Gamma_k]$-mod).

(3) *For any diagonalizable group D over k, we have a canonical isomorphism*

$$X_*(D) \simeq \mathrm{Hom}_{\mathbb{Z}}(X^*(D), \mathbb{Z})$$

of $\mathbb{Z}[\Gamma_k]$-modules.

Using this theorem, one obtains the following consequence immediately.

Corollary 2.2. *Let T be an algebraic torus over k.*

(1) *There is a natural bijection between the set of algebraic k-subtori of T and the set of quotient $\mathbb{Z}[\Gamma_k]$-modules of the character group $X^*(T)$ which are free $\mathbb{Z}$-modules.*

(2) *There is a natural bijection between the set of algebraic k-subtori of T and the set of saturated $\mathbb{Z}[\Gamma_k]$-submodules of the cocharacter group $X_*(T)$.*

A $\mathbb{Z}$-sublattice L of a $\mathbb{Z}$-lattice X is called *saturated* if the quotient abelian group X/L is torsion-free. For any $\mathbb{Z}$-sublattice L of X there is a unique maximal $\mathbb{Z}$-sublattice L' in X of the same rank that contains L; this lattice is saturated and is called *the saturation of L in X*. Clearly, L' can be constructed by putting $L' = L \cdot \mathbb{Q} \cap X$.

Let $\alpha : S \to T$ be a homomorphism of algebraic tori over $\bar{k}$. We have the induced homomorphisms

$$\alpha_* : X_*(S) \to X_*(T) \quad \text{and} \quad \alpha^* : X^*(T) \to X^*(S)$$

given by $\alpha_*(\gamma) = \alpha \circ \gamma$ and $\alpha^*\chi = \chi \circ \alpha$ for $\gamma \in X_*(S)$ and $\chi \in X^*(T)$:

$$\mathbb{G}_{\mathrm{m}} \xrightarrow{\ \gamma\ } S \xrightarrow{\ \alpha\ } T, \quad S \xrightarrow{\ \alpha\ } T \xrightarrow{\ \chi\ } \mathbb{G}_{\mathrm{m}}.$$

The natural perfect pairing $X^*(T) \times X_*(T) \to \mathbb{Z} = \mathrm{End}(\mathbb{G}_{\mathrm{m}})$ is simply the composition $\langle \chi, \gamma \rangle_T = \chi \circ \gamma$. Then we have the adjoint property:

$$\langle \chi, \alpha_*\gamma \rangle_T = \langle \alpha^*\chi, \gamma \rangle_S, \quad \forall\, \chi \in X^*(T),\ \gamma \in X_*(S). \tag{2.1}$$

Indeed, we check $\langle \chi, \alpha_*\gamma \rangle_T = \chi \circ \alpha \circ \gamma = \langle \alpha^*\chi, \gamma \rangle_S$.

2.2. **Algebraic $\mathbb{Q}$-tori.** Let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers in $\mathbb{C}$. All number fields considered in this paper are those contained in $\mathbb{C}$. For any number field K, denote by $T^K = \mathrm{Res}_{K/\mathbb{Q}}\,\mathbb{G}_{m,K}$ the associated algebraic $\mathbb{Q}$-torus. Denote by $\Sigma_K := \mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{C}) = \mathrm{Hom}(K,\overline{\mathbb{Q}})$ the set of embeddings of K; it is equipped with the $\Gamma_{\mathbb{Q}}$-action by $\sigma \cdot \phi = \sigma \circ \phi$, where $\sigma \in \Gamma_{\mathbb{Q}}$ and $\phi \in \Sigma_K$. We have an isomorphism of $\overline{\mathbb{Q}}$-algebras:

$$c : K \otimes \overline{\mathbb{Q}} \simeq \overline{\mathbb{Q}}^{\Sigma_K}, \quad a \otimes x \mapsto (\phi(a)x)_\phi.$$

The projection at the ϕ-component via the isomorphism c gives a $\overline{\mathbb{Q}}$-algebra homomorphism

$$\mathrm{pr}_\phi : K \otimes \overline{\mathbb{Q}} \simeq \overline{\mathbb{Q}}^{\Sigma_K} \to \overline{\mathbb{Q}}.$$

This defines a character and we denote this again by $\phi \in X^*(T^K)$. Clearly Σ_K forms a $\mathbb{Z}$-basis for $X^*(T^K)$. The action ${}^\sigma\phi$ is defined (as the Galois action on the set of morphisms of varieties over $\overline{\mathbb{Q}}$) by the commutative diagram

$$\begin{array}{ccc} K \otimes \overline{\mathbb{Q}} & \xrightarrow{\phi} & \overline{\mathbb{Q}} \\ {\scriptstyle 1\otimes\sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ K \otimes \overline{\mathbb{Q}} & \xrightarrow{{}^\sigma\phi} & \overline{\mathbb{Q}}. \end{array}$$

We see $({}^\sigma\phi)(a \otimes \sigma(x)) = \sigma(\phi(a)x) = \sigma\phi(x) \cdot \sigma(x)$; so ${}^\sigma\phi = \sigma\phi = \sigma \cdot \phi$ (the latter is the natural action by $\Gamma_{\mathbb{Q}}$). We obtain $X^*(T^K) = \mathbb{Z}[\Sigma_K]$, the free $\mathbb{Z}$-module generated by Σ_K, with the $\Gamma_{\mathbb{Q}}$-action by ${}^\sigma\phi = \sigma\phi$.

Let $\Sigma_K^\vee := \{\phi^\vee \ ; \ \phi \in \Sigma_K\}$ be the basis of $\mathrm{Hom}(\mathbb{Z}[\Sigma_K],\mathbb{Z})$ dual to Σ_K. As the pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$ is $\Gamma_{\mathbb{Q}}$-equivariant we have

$$\langle \phi', {}^\sigma\phi^\vee \rangle = \langle {}^{\sigma^{-1}}\phi', \phi^\vee \rangle = \begin{cases} 1, & \text{if } {}^{\sigma^{-1}}\phi' = \phi \text{ (or } \phi' = {}^\sigma\phi), \\ 0, & \text{otherwise.} \end{cases}$$

It follows that ${}^\sigma\phi^\vee = (\sigma\phi)^\vee$. So we obtain

$$X_*(T^K) = \mathbb{Z}[\Sigma_K^\vee], \quad \text{with } {}^\sigma\phi^\vee = (\sigma\phi)^\vee, \quad \forall\, \sigma \in \Gamma_{\mathbb{Q}}. \tag{2.2}$$

Let $K \subset k$ be two number fields. We have the inclusion $T^K \subset T^k$. The induced homomorphism $X^*(T^k) \to X^*(T^K)$ is given by the restriction map $\Sigma_k \to \Sigma_K$, $\widetilde{\phi} \mapsto \widetilde{\phi}|_K$. For any subset $\Phi \subset \Sigma_K$, we denote by $\widetilde{\Phi} \subset \Sigma_k$ the preimage of this restriction map. One easily checks that the cocharacter groups $X_*(T^K) \subset X_*(T^k)$ is given by the relation

$$\phi^\vee = \sum_{\widetilde{\phi}|_K = \phi} \widetilde{\phi}^\vee. \tag{2.3}$$

3. The Main Theorem of Complex Multiplication

In this section we describe the main theorem of complex multiplication due to Shimura and Taniyama [18]. Our main reference is Serre and Tate [16].

3.1. **The reflex type norms.** Let K be a CM field and $\Phi \subset \Sigma_K$ be a CM type, i.e. the complement of Φ equals its complex conjugate. Let (A, i) be an abelian variety over a number field k of CM-type (K, Φ). That is,

$$i : K \to \mathrm{End}_k^0(A) := \mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is a ring monomorphism, $2 \dim A = [K : \mathbb{Q}]$, and the character of the representation of K on the Lie algebra $\mathrm{Lie}(A_{\mathbb{C}})$ is given by $\sum_{\varphi \in \Phi} \varphi$ (In case the action i of K is defined over $\bar{k}$ instead of k, i.e. $i : K \to \mathrm{End}^0(A \otimes \bar{k})$, the pair (A, i) is said to be potentially of CM type (K, Φ)). Note that any complex abelian variety of CM type is defined over a number field [18]. Replacing k by a finite extension of k, we assume that k is Galois over $\mathbb{Q}$ and contains K. We usually identify K with its image under i, that is, we view i as the inclusion. The endomorphism algebra $\mathrm{End}_k^0(A)$ acts on $V := H_1(A(\mathbb{C}), \mathbb{Q})$ so that V becomes a free K-module of rank one. We have the inclusion $K \subset \mathrm{End}(V)$ and may regard the algebraic torus $T^K \subset \mathrm{GL}(V)$ as an algebraic subgroup over $\mathbb{Q}$.

Set

$$\Phi_k := \{\widetilde{\phi} \in \Sigma_k \ ; \ \widetilde{\phi}|_K \in \Phi\},$$

that is, $\Phi_k = \widetilde{\Phi}$ with respect to the restriction map $\Sigma_k \to \Sigma_K$. Put $H := \mathrm{Gal}(k/K)$. Let

$$H_E := \{\sigma \in \mathrm{Gal}(k/\mathbb{Q}) \ ; \ \sigma\Phi = \Phi\},$$

and let $E \subset \overline{\mathbb{Q}}$ be the fixed field of H_E, the *reflex field of the pair* (K, Φ). Notice that $\Sigma_k = \mathrm{Gal}(k/\mathbb{Q})$ is a group and that the inverse set Φ_k^{-1} is right stable under the subgroup H_E. Therefore, the set Φ_k^{-1} descends to a unique CM type $\Phi_E \subset \Sigma_E$. The norm map $N_{\Phi_k^{-1}} : k^\times \to k^\times$ given by

$$N_{\Phi_k^{-1}}(x) := \prod_{\sigma \in \Phi_k^{-1}} \sigma(x) \tag{3.1}$$

defines a homomorphism $N_{\Phi_k^{-1}} : T^k \to T^k$ of algebraic tori over $\mathbb{Q}$. It factors through the subtorus $\psi : T^k \to T^K$ and we have the following commutative diagram

$$\begin{array}{ccc} T^k & \xrightarrow{N_{\Phi_k^{-1}}} & T^k \\ {\scriptstyle N_{k/E}}\big\downarrow & & \cup \\ T^E & \xrightarrow{N_{\Phi_E}} & T^K, \end{array} \tag{3.2}$$

where $N_{k/E}$ is the usual norm map and N_{Φ_E} is the reflex norm map defined in the same manner as $N_{\Phi_k^{-1}}$ ($\psi = N_{\Phi_E} \circ N_{k/E}$). Remark that ψ is the same homomorphism constructed by the determinant map using the (K, k)-bimodule structure of the Lie algebra $\mathrm{Lie}(A/k)$ of A in Serre and Tate [16, Section 7].

Let

$$\begin{aligned} &\psi_0 : k^\times \to K^\times, \\ &\psi_\ell : k_\ell^\times \to K_\ell^\times, \\ &\psi_\infty : k_\infty^\times \to K_\infty^\times, \end{aligned} \tag{3.3}$$

be the homomorphisms induced from the morphism $\psi : T^k \to T^K$ by evaluating at $\mathbb{Q}$, $\mathbb{Q}_\ell$ and $\mathbb{R}$, respectively, where $k_\ell = \mathbb{Q}_\ell \otimes k$, $K_\ell = \mathbb{Q}_\ell \otimes K$, $k_\infty = \mathbb{R} \otimes k$ and $K_\infty = \mathbb{R} \otimes K$.

3.2. **The explicit reciprocity law.** Let V^k denote the set of all places of k, V^k_∞ (resp. V^k_f) the set of all archimedean (resp. finite) places of k, and V^k_ℓ the set of the finite places lying above ℓ. Let S_A be the finite set of finite places where A has bad reduction.

If $v \in V^k_f - S_A$, let $k(v)$, $A(v)$, $\pi_{A(v)}$ denote respectively the residue field at v, the reduction of A at v, the Frobenius endomorphism of $A(v)$ relative to $k(v)$. The reduction map $\mathrm{End}(A) \to \mathrm{End}(A(v))$ defines an injection

$$i_v : K \to \mathrm{End}^0(A) \to \mathrm{End}^0(A(v)).$$

Since $\pi_{A(v)}$ commutes with every $k(v)$-endomorphism of $A(v)$, it lies in the commutator of the image $\mathrm{Im}(i_v)$ of i_v, which is again $\mathrm{Im}(i_v)$. Thus there is an unique element $\pi_v \in K$ such that $i_v(\pi_v) = \pi_{A(v)}$; we call π_v the *Frobenius element* attached to v.

Let I_k denote the idèle group of k. For each finite set $S \subset V^k$, let $I^S_k \subset I_k$ denote the group of idèles (a_v) such that $a_v = 1$ for all $v \in S$.

The next two theorems, due to Serre and Tate [16, Theorems 10 and 11, Section 7], are reformulation of results of Shimura-Taniyama [18] and Weil [21].

Theorem 3.1. *There exists a unique homomorphism*

$$\varepsilon : I_k \to K^\times \tag{3.4}$$

satisfying the following three conditions:

(a) *The restriction of ε to $k^\times$ is the map $\psi_0 : k^\times \to K^\times$ defined in (3.3).*
(b) *The homomorphism ε is continuous, in the sense that its kernel is open in I_k.*
(c) *There is a finite set $S \subset V^k$ containing V^k_∞ and S_A such that*

$$\varepsilon(a) = \prod_{v \notin S} \pi_v^{v(a_v)}, \quad \forall\, a \in I^S_k. \tag{3.5}$$

The last condition means that, for $v \notin S$ one has $\varepsilon(\varpi_v) = \pi_v$, where ϖ_v is any uniformizer of the completion k_v at v, .

Let

$$\rho_\ell : \Gamma_k \to \mathrm{GL}(V_\ell)$$

be the ℓ-adic Galois representation associated to the Tate module $V_\ell(A)$. As the image of ρ_ℓ is contained in $K_\ell^\times$, the map ρ_ℓ factors through $\mathrm{Gal}(k^{\mathrm{ab}}/k)$, where k^{ab} is the maximal abelian extension of k. Class field theory allows us to interpret ρ_ℓ as a homomorphism

$$\rho_\ell : I_k \to K_\ell^\times$$

which is trivial on $k^\times$. If $v \notin (S_A \cup V^k_\ell)$, then ρ_v is *unramified at* v (i.e. ρ_ℓ is trivial on $\mathcal{O}_v^\times$) and $\rho_\ell(\varpi_v) = \pi_v$. (The normalization of the Artin map $I_k \to \mathrm{Gal}(k^{\mathrm{ab}}/k)$ used in Serre-Tate [16] sends each uniformizer ϖ_v to the *arithmetic* Frobenius automorphism Fr_v.)

Theorem 3.2. *For each prime number ℓ, we have*

$$\rho_\ell(a) = \varepsilon(a)\psi_\ell(a_\ell^{-1}), \quad \forall\, a \in I_k, \tag{3.6}$$

where a_ℓ denotes the component of a in $k_\ell^\times$, and $\psi_\ell : k_\ell^\times \to K_\ell^\times$ is the map defined in (3.3).

4. Proof of MTC for CM abelian varieties

We preserve the notations and hypotheses in the previous section. Let T_0 be the image of the homomorphism $\psi : T^k \to T^K$. The Mumford-Tate conjecture for the abelian variety (A, i) of CM type, i.e. $G^0_{A,\ell} = MT(A) \otimes \mathbb{Q}_\ell$, will follow from the following two lemmas.

Lemma 4.1. *We have $G^0_{A,\ell} = T_0 \otimes \mathbb{Q}_\ell$.*

Proof. By Theorem 3.2, the map ρ_ℓ agrees with ψ_ℓ^{-1} on an open subgroup U_ℓ of $k_\ell^\times$. Since U_ℓ is Zariski dense in T^k, the Zariski closure of the image $\rho_\ell(I_k)$ contains the image of ψ_ℓ (as an algebraic $\mathbb{Q}_\ell$-torus), which is $T_0 \otimes \mathbb{Q}_\ell$. This shows the inclusion $T_0 \otimes \mathbb{Q}_\ell \subset G^0_{A,\ell}$.

On the other hand, the map $\rho_\ell : I_k \to T^K(\mathbb{Q}_\ell)/T_0(\mathbb{Q}_\ell)$ factors through the quotient:

$$\rho_\ell : I_k/k^\times U \to T^K(\mathbb{Q}_\ell)/T_0(\mathbb{Q}_\ell), \tag{4.1}$$

where U is the kernel of ε. Notice $U \supset (k_\infty^\times)^0$. By the finiteness of class numbers, the group $I_k/k^\times U$ is finite. Therefore, ρ_ℓ has finite image in $T^K(\mathbb{Q}_\ell)/T_0(\mathbb{Q}_\ell)$. This shows the inclusion $G^0_{A,\ell} \subset T_0 \otimes \mathbb{Q}_\ell$.

We then conclude the equality $G^0_{A,\ell} = T_0 \otimes \mathbb{Q}_\ell$. ∎

Lemma 4.2. *We have $MT(A) = T_0$.*

Proof. The Hodge cocharacter $\mu : \mathbb{C}^\times \to (K \otimes \mathbb{C})^\times$ has the property

$$\langle \phi, \mu \rangle = \begin{cases} 1 & \text{if } \phi \in \Phi; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\mu = \sum_{\phi\in\Phi} \phi^\vee$. As $MT(A)$ is the smallest $\mathbb{Q}$-torus of T^K containing the image of μ, its cocharacter group is equal to the saturation of the sublattice

$$\mathbb{Z}\,[\,{}^\sigma\mu\ ;\ \sigma \in \mathrm{Gal}(k/\mathbb{Q})\,] \subset X_*(T^K).$$

Now we want to determine the image of the map $X_*(T^k) \to X_*(T^K) \subset X_*(T^k)$ induced by ψ. Note that the map induced by ψ is that induced by the reflex norm map $N_{\Phi_k^{-1}}$. We claim

$$\psi_*(\tau^\vee) = {}^\tau\mu \quad \text{in } X_*(T^K) = \mathbb{Z}[\Sigma_K^\vee] \subset \mathbb{Z}[\Sigma_k^\vee] \tag{4.2}$$

for every $\tau \in \Sigma_k$.

For $\sigma \in \mathrm{Gal}(k/\mathbb{Q})$, the isomorphism $\sigma : k \to k$ induces an isomorphism $\sigma : T^k \to T^k$ of algebraic $\mathbb{Q}$-tori. The pullback map $\sigma^* : X^*(T^k) \to X^*(T^k)$ is given by $\sigma^*\tau = \tau\sigma$. For the push-forward map σ_* we have $\sigma_*\tau^\vee = (\tau \circ \sigma^{-1})^\vee$ for $\tau \in \Sigma_k$; this follows from

$$\langle \tau', \sigma_*\tau^\vee \rangle = \langle \tau'\sigma, \tau^\vee \rangle = \begin{cases} 1 & \text{if } \tau'\sigma = \tau \text{ (or } \tau' = \tau\sigma^{-1}); \\ 0 & \text{otherwise.} \end{cases}$$

We compute

$$(N_{\Phi_k^{-1}})_*(\tau^\vee) = \sum_{\sigma\in\Phi_k^{-1}} (\tau\circ\sigma^{-1})^\vee = \sum_{\sigma\in\Phi_k} (\tau\circ\sigma)^\vee = \sum_{\phi\in\Phi} (\tau\circ\phi)^\vee.$$

On the other hand we also have

$$^{\tau}\mu = \sum_{\phi\in\Phi} {}^{\tau}\phi^{\vee} = \sum_{\phi\in\Phi} (\tau\circ\phi)^{\vee}.$$

This proves our claim.

It follows from our claim that the torus T_0 corresponds to the saturation of the $\mathbb{Z}$-sublattice

$$\mathbb{Z}\,[\,^{\tau}\mu\ ;\ \tau\in \mathrm{Gal}(k/\mathbb{Q})\,] \subset X_*(T^K).$$

By Corollary 2.2, one has $MT(A)=T_0$. ∎

5. The MTC for CM motives

In this section we extend Pohlmann's theorem to CM motives. It is more convenient to extend the CM field K to a CM algebra, that is, a product of CM fields. We consider abelian varieties of CM type (K,Φ), where K is a CM algebra and Φ is a CM type. As the base field k does not play a role in the MTC, we may assume that it is Galois over $\mathbb{Q}$ containing k and that the monodromy group $\rho_\ell(\Gamma_k)$ is contained in $G^0_{A,\ell}(\mathbb{Q}_\ell)$. It is easy to show the MTC for abelian varieties of CM type can be reduced to the case where K is a CM field and thus holds.

By a CM motive we mean a CM motives in André's category of motives with respect to motivated cycles. We refer to [1] or [3] for the formal definition. A result states that every CM motive is a summand of a tensor product of copies of the motive of a abelian variety of CM type (see [7, Proposition 1.1]).

Suppose M is a CM motive over a number field k. Then replacing k by a suitable finite extension if necessary, there is an abelian variety of CM type over k so that M is a summand of a tensor product of the motive $h(A)$ of A. As before, let $V=H_1(A_{\mathbb{C}},\mathbb{Q})$ and we identify $V_\ell := V\otimes\mathbb{Q}_\ell$ with $V_\ell(A)$. Write

$$V(m,n,r)=V^{\otimes m}\otimes_{\mathbb{Q}} \check{V}^{\otimes n}\otimes_{\mathbb{Q}}\mathbb{Q}(r) \text{ and } V_\ell(m,n,r)=V_\ell^{\otimes m}\otimes_{\mathbb{Q}_\ell} \check{V}_\ell^{\otimes n}\otimes_{\mathbb{Q}_\ell}\mathbb{Q}_\ell(r),$$

respectively, where m and n are nonnegative integers and $r\in\mathbb{Z}$. We have a canonical isomorphism $V(m,n,r)\otimes\mathbb{Q}_\ell \simeq V_\ell(m,n,r)$.

As a remark in the Introduction, the Tannakian category $\mathcal{T}(V)$ generated by the Hodge structure $V\oplus\mathbb{Q}(1)$ is equivalent to the Tannakian category $\mathcal{T}(V_\ell)$ generated by the $\mathbb{Q}_\ell$-representation $V_\ell\oplus\mathbb{Q}_\ell(1)$ of Γ_k. As a result, for any Hodge substructure $W\subset V(m,n,r)$, the $\mathbb{Q}_\ell$ subspace $W\otimes\mathbb{Q}_\ell\subset V(m,n,r)\simeq V_\ell(m,n,r$ is stable under the Γ_k-action. Conversely any subrepresentation W_ℓ of $V_\ell(m,n,r)$ is of the form $W\otimes\mathbb{Q}_\ell$ for a unique Hodge substructure $W\subset V(m,n,r)$.

To prove the MTC for the CM motive M, one may show that the Tannakian categories $\mathcal{T}(M_B)$ and $\mathcal{T}(M_\ell)$ are equivalent, where M_B and M_ℓ are the Betti and ℓ-adic realizations of M, respectively. Suppose that W is an object in $\mathcal{T}(M_B)$. Then $W\otimes\mathbb{Q}_\ell$ gives an object in $\mathcal{T}(M_\ell)$ since $W\subset M_B(m,n,r)$ for some integers m,n,r and hence $W\subset V(m',n',r')$ for some other integers m',n',r'. Conversely, any object $W_\ell\subset M_\ell(m,n,r)\subset V_\ell(m',n',r')$ is of the form $W\otimes\mathbb{Q}_\ell$ for a unique object $W\subset V(m',n',r')$. It follows from the uniqueness that $W\subset M_B(m,n,r)$. This proves the equivalence of $\mathcal{T}(M_B)$ and $\mathcal{T}(M_\ell)$ and hence the MTC for M.

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References

[1] Yves André, *Une introduction aux motifs (motifs purs, motifs mixtes, périod)*. Panoramas et Synthèses, 17. Société Mathématique de France, Paris, 2004. xii+261 pp.

[2] A. Borel, *Linear algebraic groups*. Second edition. Graduate Texts in Mathematics, **126**. Springer-Verlag, New York, 1991.

[3] P. Deligne, Valeurs de fonctions L et périodes d'intégrales. *Automorphic forms, representations and L-functions* Proc. Sympos. Pure Math., 33, Part 2, 313–346, 1979.

[4] P. Deligne, J. Milne, A. Ogus and K.-Y. Shih, Hodge cycles, motives, and Shimura varieties. Lecture Notes in Math., vol. 900, Springer-Verlag, 1982.

[5] G. Faltings, Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.* **73** (1983), 349–366.

[6] M. Larsen and R. Pink, Abelian varieties, ℓ-adic representations and ℓ-independence, *Math. Ann.* **302** (1995), 561–579.

[7] Arthur Ogus, A p-adic analogue of the Chowla-Selberg formula. *p-adic analysis (Trento, 1989)*, 319–341, Lecture Notes in Math., vol. 1454, Springer-Verlag, 1990.

[8] R. Pink, l-adic algebraic monodromy groups, cocharacters, and the Mumford-Tate conjecture. *J. Reine Angew. Math.* **495** (1998), 187–237.

[9] H. Pohlmann, Algebraic cycles on abelian varieties of complex multiplication type. *Ann. Math.* **88** (1968), 161–180.

[10] K.A. Ribet, Galois action on division points of abelian varieties with real multiplications, *Amer. J. Math.* **98** (1976), no. 3, 751–804.

[11] K.A. Ribet, Dividing rational points of abelian varieties of CM type, *Compositio Math.* **33** (1976), 69–74.

[12] K.A. Ribet, Division points of abelian varieties with complex multiplication, *Bull. Soc. Math. France* **2** (1980), 75–94.

[13] K.A. Ribet, Review of *Abelian l-adic representations and elliptic curves* by J.-P. Serre. *Bull. Amer. Math. Soc. (N.S.)* **22** (1990), no. 1, 214–218.

[14] J.-P. Serre, Résumé des cours de 1984-1985. pp. 27–32, Œuvres Collected papers. IV. 1985–1998, 2000.

[15] J.-P. Serre, Propriétés conjecturales des groupes de Galois motiviques et des représentations ℓ-adiques, *Motives* (Seattle, WA, 1991), *Proc. Symp. Pure Math.,* vol. **55**, Part 1, Amer. Math. Soc., Providence, RI, 1994, pp. 377–400.

[16] J.-P. Serre and J. Tate, Good reduction of abelian varieties. *Ann. Math.* **88** (1968), 492–517.

[17] G. Shimura, *Abelian varieties with complex multiplication and modular functions*. Princeton Mathematical Series, 46. Princeton University Press, Princeton, NJ, 1998.

[18] G. Shimura and Y. Taniyama, *Complex multiplication of abelian varieties and its applications to number theory*. Publ. Math. Soc. Japan 6, Tokyo, 1961.

[19] S. G. Tankeev, On the Mumford-Tate conjecture for abelian varieties. Algebraic geometry, 4. *J. Math. Sci.* **81** (1996), no. 3, 2719–2737.

[20] S. G. Tankeev, Cycles on abelian varieties and exceptional numbers, *Izv. Ross. Akad. Nauk. Ser. Mat.* **60** (1996), 159–194, translation in *Izv. Math.* **60** (1996), no. 2, 391–424.

[21] A. Weil, On a certain type of characters of the idèle-class group of an algebraic number-field. *Proceedings of the international symposium on algebraic number theory*, Tokyo & Nikko, 1955, pp. 1–7.

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